COMPUTATIONS ON SOFIC S-GAP SHIFTS

DAWOUD AHMADI DASTJERDI AND SOMAYE JANGJOO

ABSTRACT. Let $S = \{s_n\}$ be an increasing finite or infinite subset of $\mathbb{N}\bigcup\{0\}$ and $X(S)$ the S-gap shift associated to S. Let $f_S(x) = 1 - \sum \frac{1}{x^{s_n+1}}$ be the entropy function which will be vanished at $2^{h(X(S))}$ where $h(X(S))$ is the entropy of the system. Suppose $X(S)$ is sofic with adjacency matrix A and the characteristic polynomial χ_A. Then for some rational function Q_S, $\chi_A(x) = Q_S(x)f_S(x)$. This Q_S will be explicitly determined. We will show that $\zeta(t) = \frac{1}{f_S(t^{-1})}$ or $\zeta(t) = \frac{1}{(1-t)f_S(t^{-1})}$ when $|S| < \infty$ or $|S| = \infty$ respectively. Here ζ is the zeta function of $X(S)$. We will also compute the Bowen-Franks groups of a sofic S-gap shift.

INTRODUCTION

In symbolic dynamical systems on finite alphabets, *shifts of finite type* or SFT's are rich and show fairly simple behavior. These are symbolic dynamical systems which have finite forbidden blocks. Their factors, called *sofic* systems, show more or less the same properties but with some more complexity. All the sofic systems are accompanied by a matrix called *adjacency matrix* where virtually all the system's specifications can be read from that. In particular, the characteristic polynomial and various groups associated to the adjacency matrix carry many information about the behavior of the system. Amongst the classical examples of symbolic dynamical systems, S-gap shifts are also easy to define and grasp: fix an increasing subset S of $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. If S is finite, define $X(S)$, the space of S-gap shift associated to S to be the set of all binary sequences for which 1's occur infinitely often in each direction and such that the number of 0's between successive occurrences of a 1 is an integer in S. When S is infinite, we need to allow points that begin or end with an infinite string of 0's. The richness of the non-trivial sofic systems, their simplicity combined with those of S-gap shifts make the sofic S-gap shifts a valuable source in application such as coding of data and in theory as a source for constructing examples to identify the universal behaviors in symbolic dynamics.

In this paper, we consider the sofic S-gap shifts and we will introduce the notations and some backgrounds in section 1, then in section 2 we will introduce the function

$$f_S(x) = 1 - \sum_{s_n \in S} \frac{1}{x^{s_n+1}} \tag{0.1}$$

and the relation between this function and the characteristic polynomial will be revealed. In fact, $f_S(x)$ is a map that is used to compute the entropy of $X(S)$ [7]. Also we will compute the zeta function of sofic S-gap shifts in terms of entropy

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function. The last section is devoted to compute the Bowen-Franks groups and stating the related results.

1. Background and Notations

The notations has been taken from [7] and the proofs of the claims in this section can be found there. Let $\mathcal{A}$ be an alphabet, that is a nonempty finite set. The full $\mathcal{A}$-shift denoted by $\mathcal{A}^{\mathbb{Z}}$, is the collection of all bi-infinite sequences of symbols from $\mathcal{A}$. A block (or word) over $\mathcal{A}$ is a finite sequence of symbols from $\mathcal{A}$. The shift function σ on the full shift $\mathcal{A}^{\mathbb{Z}}$ maps a point x to the point $y = \sigma(x)$ whose ith coordinate is $y_i = x_{i+1}$.

Let $\mathcal{B}_n(X)$ denote the set of all admissible n blocks. The *Language* of X is the collection $\mathcal{B}(X) = \bigcup_{n=0}^{\infty} \mathcal{B}_n(X)$. A word $v \in \mathcal{B}(X)$ is *synchronizing* if whenever uv and vw are in $\mathcal{B}(X)$, we have $uvw \in \mathcal{B}(X)$. An irreducible shift space X is a *synchronized system* if it has an synchronizing word [3].

An edge shift, denoted by X_G, is a shift space which consist of all bi-infinite walks in a directed graph G.

A labeled graph $\mathcal{G}$ is a pair $(G, \mathcal{L})$ where G is a graph with edge set $\mathcal{E}$, and the labeling $\mathcal{L} : \mathcal{E} \to \mathcal{A}$. A *sofic shift* $X_{\mathcal{G}}$ is the set of sequences obtained by reading the labels of walks on G,

$$X_{\mathcal{G}} = \{\mathcal{L}_\infty(\xi) : \xi \in X_G\} = \mathcal{L}_\infty(X_G).$$

We say $\mathcal{G}$ is a *presentation* of $X_{\mathcal{G}}$. Every SFT is sofic, but the converse is not true.

A labeled graph $\mathcal{G} = (G, \mathcal{L})$ is *right-resolving* if for each vertex I of G the edges starting at I carry different labels. A *minimal right-resolving presentation* of a sofic shift X is a right-resolving presentation of X having the fewer vertices among all right-resolving presentations of X.

Let X be a shift space and $w \in \mathcal{B}(X)$. The *follower set* $F(w)$ of w is defined by $F(w) = \{v \in \mathcal{B}(X) : wv \in \mathcal{B}(X)\}$. A shift space X is sofic if and only if it has a finite number of follower sets [7, Theorem 3.2.10]. In this case, we have a labeled graph $\mathcal{G} = (G, \mathcal{L})$ called the *follower set graph* of X. The vertices of G are the follower sets and if $wa \in \mathcal{B}(X)$, then draw an edge labeled a from $F(w)$ to $F(wa)$. If $wa \notin \mathcal{B}(X)$ then do nothing.

The entropy of a shift space X is defined by $h(X) = \lim_{n\to\infty}(1/n) \log |\mathcal{B}_n(X)|$ where $\mathcal{B}_n(X)$ is the set of all admissible n blocks.

2. Entropy Function, Characteristic Polynomial and Zeta Function

Let $S = \{s_n\}_n$ be an increasing sequence in $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and let $d_0 = s_0$ and $\Delta(S) = \{d_n\}_n$ where $d_n = s_n - s_{n-1}$. In [1], it has been proved that $X(S)$ is a subshift of finite type (SFT) if and only if S is finite or cofinite and it is sofic if and only if $\Delta(S)$ is eventually periodic.

2.1. **Adjacency Matrix.** Let $X(S)$ be a sofic shift. It is an easy exercise to see that the minimal right-resolving presentation of $X(S)$ is the labeled subgraph of the follower set graph consisting of only the follower sets of synchronizing words. For S-gap shifts, 1 is a synchronizing word. Thereby for all $i \in \mathbb{N}$, 10^i is also a synchronizing word [7, Lemma 3.3.15] and $F(w) = F(1)$ for $w = w_1w_2...w_q \in \mathcal{B}(X(S))$ for which $w_q = 1$. Furthermore, 0^i, $i \in \mathbb{N}$, is not a synchronizing word. So if $S = \{s_1, s_2, ..., s_k\}$ is a finite subset of $\mathbb{N}_0$, then the adjacency matrix of $X(S)$ will be an $(s_k + 1) \times (s_k + 1)$ matrix

$$A = \begin{pmatrix} a_{11} & 1 & 0 & 0 & \dots & 0 \\ a_{21} & 0 & 1 & 0 & \dots & 0 \\ a_{31} & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{s_k 1} & 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 \end{pmatrix} \tag{2.1}$$

where $a_{i1} = 1$, $i \in \{s_1 + 1, ..., s_k + 1\}$ and for other i's $a_{i1} = 0$. When S is infinite, $\Delta(S)$ is eventually periodic [1] and we set

$$\Delta(S) = \{d_1, d_2, ..., d_k, \overline{g_1, g_2, ..., g_l}\} \tag{2.2}$$

where $g_i = s_{k+i} - s_{k+i-1}, 1 \leq i \leq l$ and $s_0 = 0$. We use $\overline{g_1, g_2, ..., g_l}$ to show that $g_1, g_2, ..., g_l$ repeat forever. In this situation, three cases arises where for each case the adjacency matrix of $X(S)$ will be as follow:

$$A = \begin{pmatrix} a_{11} & 1 & 0 & \dots & 0 \\ a_{21} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{(n-1)1} & 0 & 0 & \dots & 1 \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{pmatrix} \tag{2.3}$$

here $a_{i1} = 1$, $i \in \{s_1 + 1, ..., s_{k+l-1} + 1\}$ and for other i's, $a_{i1} = 0$ and $\sum_{j=2}^{n} a_{nj}$ is either zero or 1.

2.1.1. If $k = 1$ and $g_l > s_1$, then

(1) for $g_l = s_1 + 1$, $F(10^{s_l+1}) = F(1)$ and the adjacency matrix will be 2.3 for $n = s_l + 1$, $a_{n1} = 2$ and $a_{nj} = 0$, $2 \leq j \leq n$.
(2) for $g_l > s_1 + 1$, $F(10^g) = F(1)$ and the adjacency matrix will be 2.3 for $n = g$, $a_{n1} = 1$ and $a_{nj} = 0$, $2 \leq j \leq n$.

2.1.2. For $k \neq 1$, if $g_l > d_k$, then $F(10^{g+s_{k-1}+1}) = F(10^{s_{k-1}+1})$. So the adjacency matrix will be 2.3 for $n = g + s_{k-1} + 1$ and $a_{n(s_{k-1}+2)} = 1$.

2.1.3. For $k \in \mathbb{N}$, if $g_l \leq d_k$, then $F(10^{s_{k+l-1}+1}) = F(10^{s_k - g_l + 1})$. So the adjacency matrix will be 2.3 for $n = s_{k+l-1} + 1$ and $a_{n1} = a_{n(s_k - g_l + 2)} = 1$. This includes the case when $k = 1$ and $g_l \leq s_1$.

2.2. **Characteristic Polynomial in terms of Entropy Function.** The entropy of an S-gap shift is $\log \lambda$ where λ is a unique nonnegative solution of the $\sum_{n \in S} x^{-(n+1)} = 1$ [9]. Call $f_S(x)$ in 0.1 the *entropy function*; and note that $f_S(2^{h(X(S))}) = 0$ where $h(X(S))$ is the entropy of $X(S)$. Suppose $X(S)$ is a sofic shift with the adjacency matrix A. Our goal is to compute, χ_A the characteristic polynomial of A explicitly by using elements of S.

The following lemma computes the determinant of a special matrix which will be used several times. The proof is elementary and straightforward; one needs only compute the determinant on the expansion of the first row.

Lemma 2.1. *Let E be an $n \times n$ matrix with $e_{i1} = -1$, $i \in \{\ell_1, \ell_2, ..., \ell_m\} \subseteq \{1, 2, ..., n\}$. Also, for $i \neq j$,*

$$e_{ij} = \begin{cases} -1 & j = i+1, \\ 0 & \text{otherwise.} \end{cases}$$

Then if $\ell_m < n$,

$$\det E = -\prod_{\ell_1+1}^{n} e_{ii} - \prod_{\ell_2+1}^{n} e_{ii} - \ldots - \prod_{\ell_m+1}^{n} e_{ii}.$$

If $\ell_m = n$, then the last term above will be -1, or equivalently

$$\det E = -\prod_{\ell_1+1}^{n} e_{ii} - \prod_{\ell_2+1}^{n} e_{ii} - \ldots - \prod_{\ell_{m-1}+1}^{n} e_{ii} - 1.$$

Theorem 2.2. *Let $X(S)$ be a sofic shift with the adjacency matrix A.*

(1) *Suppose $X(S)$ is an SFT shift.*
   (a) *If $S = \{s_1, s_2, ..., s_k\}$ is a finite subset of $\mathbb{N}_0$, then*
   $$\chi_A(x) = x^{s_k+1} f_S(x).$$
   (b) *If S is cofinite (let $\Delta(S) = \{d_1, d_2, ..., d_k, \overline{1}\}$), then*
   $$\chi_A(x) = x^{s_k}(x-1) f_S(x).$$

(2) *Suppose $X(S)$ is strictly sofic.Then $\Delta(S)$ is eventually periodic* [1]. *Let*
   $$\Delta(S) = \{d_1, d_2, ..., d_k, \overline{g_1, g_2, ..., g_l}\}$$
   where $g_i = s_{k+i} - s_{k+i-1}, 1 \leq i \leq l$; and set $g = \sum_{i=1}^{l} g_i$.
   (a) *For $k = 1$, $g_l > s_1$,*
   $$\chi_A(x) = (x^g - 1) f_S(x).$$
   (b) *For $g_l \leq d_k$,*
   $$x^g \chi_A(x) = x^{s_{k+l-1}+1}(x^g - 1) f_S(x).$$
   (c) *For $k \neq 1$, $g_l > d_k$,*
   $$x^{s_{k+l-1} - s_{k-1}} \chi_A(x) = x^{s_{k+l-1}+1}(x^g - 1) f_S(x).$$

Proof. We prove the theorem for $0 \notin S$ and at the end a comment will be given to see how the theorem works for $0 \in S$. To compute the characteristic polynomial of A, we consider different cases and in all cases, we use the expansion with the first row of

$$B = x\mathrm{Id} - A \tag{2.4}$$

to obtain the determinant. Let B_{ij} be the first minor matrix associated to (i, j), that is the sub-matrix obtained by deleting ith row and jth column of B.

Suppose $X(S)$ is an SFT shift and $S = \{s_1, s_2, ..., s_k\}$. Then

$$f_S(x) = 1 - \left(\frac{1}{x^{s_1+1}} + \ldots + \frac{1}{x^{s_k+1}} \right).$$

Let A be as in 2.1. Since $0 \notin S$, then $a_{11} = 0$ and we have

$$\chi_A(x) = x \det(B_{11}) + \det(B_{12}) = x(x^{s_k}) + \det(B_{12}).$$

Applying Lemma 2.1 for $E = B_{12}$ gives the conclusion for the case $|S| < \infty$. That is,

$$\chi_A(x) = x(x^{s_k}) + (-x^{s_k - s_1} - ... - x^{s_k - s_{k-1}} - 1) = x^{s_k+1} f_S(x).$$

Now suppose S is cofinite, that is $\Delta(S) = \{d_1, d_2, ..., d_k, \overline{1}\}$. We have

$$f_S(x) = 1 - \frac{1}{x^{s_1+1}} - \dots - \frac{1}{x^{s_{k-1}+1}} - \frac{1}{x^{s_k+1}} \left(\frac{x}{x-1} \right).$$

Apply 2.1.3 to have adjacency matrix A in 2.3 for $n = s_k + 1$ and $a_{nn} = 1$. Then set $E = B_{12}$, $\ell_i = s_i + 1$ and $m = k$ in Lemma 2.1 to have

$$\begin{aligned} \chi_A(x) &= x(x^{s_k-1}(x-1)) + (-x^{(s_k-s_1)-1} - \dots - x^{(s_k-s_{k-1})-1})(x-1) - 1 \\ &= x^{s_k}(x-1) f_S(x). \end{aligned}$$

That is,

$$\chi_A(x) = x^{s_k}(x-1) f_S(x).$$

Now suppose $X(S)$ is an strictly sofic shift . Then

(2.5)

$$f_S(x) = 1 - \left(\frac{1}{x^{s_1+1}} + \dots + \frac{1}{x^{s_{k-1}+1}} \right) - \left(\frac{1}{x^{s_k+1}} + ... + \frac{1}{x^{s_{k+l-1}+1}} \right) \left(\frac{x^g}{x^g - 1} \right).$$

Our goal is to find $\det(B) = \det(x\mathrm{Id} - A)$. So we consider several cases, and in all cases there is a $(p+g) \times (p+g)$ square sub-matrix G in the lower right corner of A:

$$G = \begin{pmatrix} x & -1 & 0 & \dots & 0 & 0 \\ 0 & x & -1 & \dots & 0 & 0 \\ 0 & 0 & x & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & x & -1 \\ 0 & 0 & \dots & -1 & \dots & x \end{pmatrix}.$$

Since $\det(G_{12}) = 0$, so $\det G = x^p \det G_g$ where

$$G_g = \begin{pmatrix} x & -1 & \dots & 0 \\ 0 & x & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & \dots & x \end{pmatrix}$$

is the lower right sub-matrix of G and $g = \sum_{i=1}^{l} g_i$. But G_g is the adjacency matrix of an S'-gap shift with $S' = \{g - 1\}$. So

$$\det G = x^p(x^g - 1). \tag{2.6}$$

Note that $b_{ii} = x$ and $b_{i(i+1)} = b_{(s_i+1)1} = -1$. In the process of computing determinant of sub-matrices if $2 \le (s_i + 1) \le n - g$, then the identity 2.6 will be used; otherwise we apply Lemma 2.1. First let $k = l = 1$.

(1) $g_1 \le s_1$. Let $n = s_1 + 1$ and use 2.1.3 to obtain the adjacency matrix A in 2.3 with $a_{n1} = a_{n(s_1-g_1+2)} = 1$ and $a_{i1} = 0$, $1 \le i \le n-1$. The matrix B_{11} associated to b_{11} is the matrix G for $p = s_1 + 1 - g_1$. So by 2.6 and Lemma 2.1,

$$\chi_A(x) = x^{s_1+1-g_1}(x^{g_1} - 1) - 1.$$

Therefore,

$$x^{g_1} \chi_A(x) = x^{s_1+1}(x^{g_1} - 1) f_S(x).$$

(2) $g_1 > s_1$. If $g_1 = s_1 + 1$, then use 2.1.1 to have
$$\chi_A(x) = x^{s_1+1} - 2 = (x^{g_1} - 1) f_S(x);$$
and if $g_1 > s_1 + 1$, use 2.1.1 and Lemma 2.1 to have
$$\chi_A(x) = x^{g_1} - x^{g_1 - s_1} - 1 = (x^{g_1} - 1) f_S(x).$$

Now suppose $\Delta(S) = \{s_1, \overline{g_1, g_2, ..., g_l}\}$. Then by an induction argument on l if $g_l \leq s_1$, we will have
$$x^g \chi_A(x) = x^{s_l+1}(x^g - 1) f_S(x);$$
and if $g_l > s_1$,
$$\chi_A(x) = (x^g - 1) f_S(x).$$
To have the general case, suppose $k > 1$. Then by an induction on k, the proof will be established. First let $\Delta(S) = \{d_1, d_2, \overline{g_1, g_2, ..., g_l}\}$ and consider two cases.

(1) $g_l \leq s_2 - s_1$. Then $n = s_{l+1} + 1$ and A will be obtained by 2.1.3. We have $\det B = x \det B_{11} + \det B_{12}$. The sub-matrix B_{11} is the matrix G with $p = s_{l+1} + 1 - g$. So by 2.6 its determinant is $x^{s_{l+1}+1-g}(x^g - 1)$. Also $\det B_{12}$ is equivalent to determinant of a sub-matrix
$$C = \begin{pmatrix} -1 & -1 & 0 & \dots & 0 & 0 \\ 0 & x & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & 0 & \dots & -1 & \dots & x \end{pmatrix}.$$
of B after deleting rows $1, 2, ..., s_1$ and columns $2, 3, ..., s_1 + 1$. This means $\det(B_{12}) = -x^{s_{l+1}-s_1-g}(x^g - 1) - (x^{s_{l+1}-s_2} + \cdots + x^{s_{l+1}-s_l} + 1)$. Therefore,
$$x^g \chi_A(x) = x^{s_{l+1}+1}(x^g - 1) f_S(x).$$

(2) $g_l > s_2 - s_1$ and $n = g + s_1 + 1$. The adjacency matrix A can be derived by 2.1.2. Therefore, similar to the above
$$\chi_A(x) = x^{s_1+1}(x^g - 1) + \det \begin{pmatrix} -1 & -1 & \dots & \dots & 0 & 0 \\ 0 & x & \dots & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & 0 & \dots & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & -1 & \dots & \dots & 0 & x \end{pmatrix}. \tag{2.7}$$
The right matrix is a $(g+1) \times (g+1)$ matrix. Note that in this matrix the -1 in the first column appears in the first row and rows $s_2 - s_1$, $s_3 - s_1$, ... and $s_{l+1} - s_1$. Therefore, Lemma 2.1 can be applied to have
$$\chi_A(x) = x^{s_1+1}(x^g - 1) - (x^g - 1) - (x^{s_1-s_2} + \cdots + x^{s_1-s_{l+1}})x^g$$
or
$$x^{s_{l+1}-s_1} \chi_A(x) = x^{s_{l+1}+1}(x^g - 1) f_S(x).$$
Now suppose $\Delta(S)$ is in the general form, that is $\Delta(S) = \{d_1, d_2, ..., d_k, \overline{g_1, g_2, ..., g_l}\}$. Similar argument gives the following

(1) If $g_l \leq d_k$, then
$$x^g \chi_A(x) = x^{s_{k+l-1}+1}(x^g - 1) f_S(x);$$
(2) If $g_l > d_k$ $(k \neq 1)$, then
$$x^{s_{k+l-1}-s_{k-1}} \chi_A(x) = x^{s_{k+l-1}+1}(x^g - 1) f_S(x).$$

To end the proof suppose $0 \in S$. Then $a_{11} = 1$ and $b_{11} = x - 1$. This means in all formulas an x must be replaced by $x-1$ or simply by multiplying by $\frac{(x-1)}{x}$. $\square$

2.3. **Zeta Function in terms of Entropy Function.** For a dynamical system (X, T), let $p_n(T)$ be the number of periodic points in X having period n. When $p_n(T) < \infty$, the zeta function ζ_T is defined as
$$\zeta_T(t) = \exp\left(\sum_{n=1}^{\infty} \frac{p_n(T)}{n} t^n\right).$$

Theorem 2.3. *Let ζ_{σ_S} be the zeta function for $X(S)$. Then $\zeta_{\sigma_S}(t)$ is either $\frac{1}{f_S(t^{-1})}$ or $\frac{1}{(1-t)f_S(t^{-1})}$ for $|S| < \infty$ or $|S| = \infty$ respectively.*

Proof. If A is an $r \times r$ nonnegative integer matrix, then
$$\zeta_{\sigma_A}(t) = \frac{1}{t^r \chi_A(t^{-1})} = \frac{1}{\det(\mathrm{Id} - tA)} \tag{2.8}$$
[7, Theorem 6.4.6]. Recall that $X(S)$ is SFT if and only if S is finite or cofinite [1]. So by 2.8, 2.5 and Theorem 2.2, if S is finite, then
$$\zeta_{\sigma_S}(t) = \frac{1}{f_S(t^{-1})} = \frac{1}{1 - t^{s_1+1} - \cdots - t^{s_k+1}}$$
and if S is cofinite,
$$\zeta_{\sigma_S}(t) = \frac{1}{(1-t)f_S(t^{-1})} = \frac{1}{(1 - t^{s_1+1} - \cdots - t^{s_{k-1}+1})(1-t) - t^{s_k+1}}.$$

Now suppose $X(S)$ is strictly sofic with $\Delta(S)$ as in (2.2). By the minimal right-resolving presentation of $X(S)$ discussed in subsection (2.1), if $n = gk$ $(k \in \mathbb{N})$, then every point in $X_{\mathcal{G}}$ of period n is the image of exactly one point in X_G of the same period, except 0^∞, which is the image of g points of period n. Hence in this case $p_n(\sigma_{\mathcal{G}}) = p_n(\sigma_G) - (g-1)$. When g does not divide n, 0^∞ is not the image of any point in X_G with period n and so $p_n(\sigma_{\mathcal{G}}) = p_n(\sigma_G) + 1$. Therefore,
$$\begin{aligned}
\zeta_{\sigma_S}(t) &= \exp\left(\sum_{\substack{n=1\\ g\nmid n}}^{\infty} \frac{p_n(\sigma_G)+1}{n} t^n + \sum_{\substack{n=1\\ g\mid n}}^{\infty} \frac{p_n(\sigma_G)-(g-1)}{n} t^n\right)\\
&= \exp\left(\sum_{n=1}^{\infty} \frac{p_n(\sigma_G)}{n} t^n + \sum_{n=1}^{\infty} \frac{t^n}{n} - g\sum_{\substack{n=1\\ g\mid n}}^{\infty} \frac{t^n}{n}\right)\\
&= \zeta_{\sigma_A}(t) \times \frac{1-t^g}{1-t}
\end{aligned}$$
and by 2.8, 2.5 and Theorem 2.2,
$$\zeta_{\sigma_A}(t) = \frac{1}{(1-t^g)f_S(t^{-1})} = \frac{1}{(1 - \sum_{i=1}^{k-1} t^{s_i+1})(1-t^g) - \sum_{i=k}^{k+l-1} t^{s_i+1}}.$$

□

Remark 2.4. One should not expect to arrive at a formula for strictly sofic case by a limiting process on SFT 's. This cannot happen even for an SFT when $|S| = \infty$. For instance, consider $S = \{1, 2, ...\}$.Then by Theorem 2.3,

$$\zeta_{\sigma_S}(t) = \frac{1}{1 - t - t^2}.$$

On the other hand, suppose $S_n = \{1, 2, ..., n\}$ for all $n \in \mathbb{N}$. Then $S_n \nearrow S$ and

$$\zeta_{\sigma_{S_n}}(t) = \frac{1}{1 - t^2 - \ldots - t^{n+1}}.$$

With an easy computation for $|t| < 1$,

$$\zeta_{\sigma_{S_n}}(t) = \frac{1}{1 - t^2 - \ldots - t^{n+1}} \to \frac{1 - t}{1 - t - t^2} \neq \zeta_{\sigma_S}(t).$$

However, $\frac{1-t}{1-t-t^2}$ is the zeta function of a mixing almost finite type shift. For let $\Lambda_1 = \{1\}$ and $\Lambda_2 = \{\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\}$. Also for any set $\Lambda \subseteq \mathbb{C}\backslash\{0\}$ define $tr_n(\Lambda) = \sum_{d|n} \mu(\frac{n}{d}) tr(\Lambda^d)$ where μ is the Möbius function. Then $tr_1(\Lambda_1) = tr_1(\Lambda_2) = 1$, $tr_n(\Lambda_1) = 0$ and $tr_n(\Lambda_2) \geq 0$ for $n \geq 2$. Hence Λ_1 and Λ_2 satisfy all the conditions given in [2, Theorem 6.1].

3. The Bowen-Franks Groups

Let A be an $n \times n$ integer matrix. The *Bowen-Franks group* of A is

$$BF(A) = \mathbb{Z}^n / \mathbb{Z}^n(\mathrm{Id} - A),$$

where $\mathbb{Z}^n(\mathrm{Id} - A)$ is the image of $\mathbb{Z}^n$ under the matrix $\mathrm{Id} - A$ acting on the right. In order to compute the Bowen-Franks group, we will use the Smith form defined below for an integral matrix. Define the *elementary operations* over $\mathbb{Z}$ on integral matrices to be:

(1) Exchanging two rows or two columns.
(2) Multiplying a row or column by -1.
(3) Adding an integer multiple of one row to another row, or of one column to another column.

Every integral matrix can be transformed by a sequence of elementary operations over $\mathbb{Z}$ into a diagonal matrix

$$\begin{pmatrix} d_1 & 0 & 0 & \ldots & 0 \\ 0 & d_2 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & d_n \end{pmatrix}$$

where $d_j \geq 0$ and d_j divides d_{j+1}. This is called the *Smith form* of the matrix [8]. If we put $\mathrm{Id} - A$ into its Smith form, then

$$BF(A) \simeq \mathbb{Z}_{d_1} \oplus \mathbb{Z}_{d_2} \oplus ... \oplus \mathbb{Z}_{d_n}.$$

By our convention, each summand with $d_j = 0$ is $\mathbb{Z}$, while summands with $d_j > 0$ are finite cyclic groups. Since $\mathbb{Z}_1$ is the trivial group, the elementary divisors of $BF(A)$ are the diagonal entries of the Smith form of $\mathrm{Id} - A$ which are not 1.

Note that $BF(A)$ (or denoted by $BF_0(A)$ in some papers) is the cokernel of $\mathrm{Id} - A$ acting on the row space $\mathbb{Z}^n$. The kernel is another Bowen-Franks group $BF_1(A) := \mathrm{Ker}(\mathrm{Id} - A)$. Similarly, acting on the column space $(\mathbb{Z}^n)^t$, $\mathrm{Id} - A$ defines another two groups as cokernel and kernel, denoted by $BF^t(A)$ and $BF^t_1(A)$ respectively. These four groups are called BF-groups [5].

Two subshifts are flow equivalent if they have topologically equivalent suspension flows [7]. Franks in [4] classified irreducible SFT's up to flow equivalent by showing that two nontrivial irreducible SFT's X_A and X_B are flow equivalent if and only if $BF(A) \simeq BF(B)$ and $\mathrm{sgn}(\det(\mathrm{Id} - A)) = \mathrm{sgn}(\det(\mathrm{Id} - B))$.

Let $X(S)$ be an SFT shift with the adjacency matrix A. The next theorem computes the Bowen-Franks groups of A.

Theorem 3.1. *Let $X(S)$ be an SFT shift with $|S| > 1$.*

(1) *Suppose $|S| < \infty$. Then $BF(A) \simeq \mathbb{Z}_{(|S|-1)} \simeq BF^t(A)$.*
(2) *Suppose $|S| = \infty$. Then $BF(A) \simeq \mathbb{Z}_1 \simeq BF^t(A)$.*
(3) *$BF_1(A) = BF^t_1(A) = \{0\}$.*

Proof. By definition of matrices in 2.1 and 2.3 and using 2.1.3, $BF_1(A) = BF^t_1(A) = \{0\}$. Let A be the adjacency matrix with the rows $\{r_1, r_2, ..., r_n\}$ and the columns $\{c_1, c_2, ..., c_n\}$. For computing $BF(A)$, first suppose $S = \{s_1, s_2, ..., s_k\}$ is finite. The following sequence of elementary operations over $\mathbb{Z}$ for $p = n$ and $q = k$ puts $\mathrm{Id} - A$ into its Smith form. We do these operations in order, that is, we first do (1) and we apply (2) to the obtained matrix in (1) and so on. In the course of operations, we call the last matrix $D = [d_{ij}]$.

(1) $c_i + c_{i+1} \to c_{i+1}$, $2 \leq i \leq n-1$. This must be done in order: first do $i = 2$ and when done do $i = 3, \cdots$.
(2) $r_i + r_1 \to r_1$, $2 \leq i \leq p$. Now $d_{11} = 1 - |S|$ and $d_{1j} = 0$ for other j's. Also all other elements in the main diagonal of D are 1 and $d_{(s_i+1)1} = -1$. Furthermore, if $0 \notin S$, then $d_{(s_1+1)1} = -1$, too.
(3) $c_{s_i+1} + c_1 \to c_1$, $1 \leq i \leq q$.
(4) $-c_1 \to c_1$.
(5) $r_1 \leftrightarrow r_n$.
(6) $c_1 \leftrightarrow c_n$.

When S is cofinite, put $p = n - 1$ and $q = k - 1$ and do the operations (1), (2) and (3) above except that if $0 \in S$, set $2 \leq i$ in the operation (3). We need the following extra operations as well.
(4$'$) Here we aim to apply an operation to have the first element in the first row zero. So $(-k + 2)c_n + c_1 \to c_1$.
(5$'$) $r_1 \leftrightarrow r_n$.
(6$'$) $-c_1 \to c_1$ and $-c_n \to c_n$. By this $BF(A)$ for all cases has been computed.

For computing $BF^t(A)$, notice that $BF^t(A)$ can be naturally identified with $BF(A^t)$, where A^t is the transpose of A. If $S = \{s_1, s_2, ..., s_k\}$ is finite, the following sequence of elementary operations over $\mathbb{Z}$ for $p = n$ puts $\mathrm{Id} - A^t$ into its Smith form.
($t1$) $c_i + c_{i+1} \to c_i$, $1 \leq i \leq p - 1$. This must be done in order: first $i = 1$, then $i = 2$, ...
($t2$) Now we have a matrix such that $d_{ii} = 1$, $d_{i(i-1)} = d_{i(i-2)} = -1$, and all other entries are zero except some on the first row. By operation

$c_i + c_{i-2} \to c_{i-2}$, $3 \le i \le p$, $d_{51} = -1$ and $c_i + c_{i-2^2} \to c_{i-2^2}$, $5 \le i \le p$, $d_{51} = 0$, but $d_{71} = -1$. In general, we let $k = \max\{l : 2^l < n\}$ and do $c_i + c_{i-2^j} \to c_{i-2^j}$ for $1 \le j \le k$, $2^j + 1 \le i \le p$.
$(t3)$ $d_{1i}r_i + r_1 \to r_1$, $2 \le i \le p$.

Now continue with the elementary operations (4), (5) and (6) above to get the result.

When S is cofinite, put $p = n - 1$ in the above and carry out with the following extra operations.
$(t4)$ $r_i + r_n \to r_n$, $2 \le i \le n - 1$.
$(t5)$ $d_{11}r_n + r_1 \to r_1$.
$(t6)$ $r_1 \leftrightarrow r_n$. This will set $d_{11} = d_{nn} = -1$.
$(t7)$ $-c_1 \to c_1$ and $-c_n \to c_n$.

□

By a different approach, the conclusion of the following corollary was reported in [6].

Corollary 3.2. *Suppose $X(S)$ is an SFT S-gap with $|S| > 1$.*

1. *If $|S| = k$, then $X(S)$ is flow equivalent to full k-shift.*
2. *If $|S| = \infty$, then $X(S)$ is flow equivalent to full 2-shift.*

Proof. First suppose X is a full k-shift. Then the adjacency matrix of X is a 1×1 matrix with $a_{11} = k$. So $BF(X) \simeq \mathbb{Z}_{k-1}$ and $\mathrm{sgn}(\det(\mathrm{Id} - \mathrm{A})) < 0$.

Now note that two nontrivial irreducible SFT's X_A and X_B are flow equivalent if and only if $BF(A) \simeq BF(B)$ and $\mathrm{sgn}(\det(\mathrm{Id} - A)) = \mathrm{sgn}(\det(\mathrm{Id} - B))$ [4]. We show that $\mathrm{sgn}(\det(\mathrm{Id} - A))$ is fixed for all SFT S-gap shifts. Then Theorem 3.1 completes the proof.

Let S be finite. The operations (4) to (6) in Theorem 3.1 change the determinant of matrix $\mathrm{Id} - A$. By these operations, $\mathrm{Id} - A$ is transformed into a diagonal matrix where its determinant is $|S| - 1$. Therefore, $\det(\mathrm{Id} - A) < 0$.

When S is cofinite, by applying operation $(5^{'})$ in Theorem 3.1, the determinant of the Smith form of A will change to $-\det(\mathrm{Id} - A)$; but the determinant of Smith form of A is 1. So $\det(\mathrm{Id} - A) < 0$. □

Suppose $X(S)$ is a proper sofic shift with the adjacency matrix A for its minimal right-resolving presentation. The following theorem computes the Bowen-Franks group of X_A. Note that $BF(X_A)$ and the $\mathrm{sgn}(\det(\mathrm{Id} - A))$ is a flow invariant but is not a complete invariant of flow equivalence of sofic S-gap shifts [6].

Theorem 3.3. *Let $X(S)$ be a proper sofic shift and let A be the adjacency matrix of minimal right-resolving presentation of $X(S)$ with $\Delta(S)$ as in 2.2. Then $BF(A) \simeq \mathbb{Z}_l \simeq BF^t(A)$ and $BF_1(A) = BF_1^t(A) = \{0\}$. Also $\det(\mathrm{Id} - A) < 0$.*

Proof. Computations uses the same routines as Theorem 3.1; so we only compute $BF(A)$. Also $\det(\mathrm{Id} - A) < 0$ can be determined by the same process as in the last corollary.

Use the notations given in the proof of Theorem 3.1 and first suppose $0 \notin S$. Let $k = 1$ and $g_l > s_1$. Put $p = n$ and do the operations (1) and (2) of that theorem but for (3) we need

(3) For $g_l = s_1 + 1$, do $c_{s_i+1} + c_1 \to c_1$, $1 \le i \le l - 1$ and $2c_n + c_1 \to c_1$. For $g_l > s_1 + 1$, do $c_{s_i+1} + c_1 \to c_1$, $1 \le i \le l$.

Now continue with (4) to (6) as in Theorem 3.1.

In the case $g_l \leq d_k$, $k \in \mathbb{N}$ and $g_l > d_k$, $k > 1$, put $p = n - 1$ and do the operations (1) and (2). We need the following extra operations for $q = k + l - 2$ and $u = s_k - g_l + 2$ when $g_l \leq d_k$ and for $q = k + l - 1$ and $u = s_{k-1} + 2$ when $g_l > d_k$.

$(3^{'})$ $c_{s_i+1} + c_1 \rightarrow c_1$, $1 \leq i \leq q$. Since $a_{n(s_k - g_l + 2)} = 1$ for $g_l \leq d_k$ and $a_{n(s_{k-1}+2)} = 1$ for $g_l > d_k$, we have $d_{n1} = -l$.

$(4^{'})$ $r_i + r_n \rightarrow r_n$, $u \leq i \leq n - 1$.

$(5^{'})$ $d_{11}c_n + c_1 \rightarrow c_1$.

$(6^{'})$ $-c_1 \rightarrow c_1$ and $-c_n \rightarrow c_n$.

$(7^{'})$ $c_1 \leftrightarrow c_n$.

Now if $0 \in S$, set $2 \leq i$ in the operations (3) and $(3^{'})$ above. $\square$

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Faculty of Mathematics ,The University of Guilan

E-mail address: `dahmadi1387@gmail.com, sjangjoo90@gmail.com`